

Global Charges of Stationary Non-Abelian Black Holes

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We consider stationary axially symmetric black holes in $SU(2)$ Einstein-Yang-Mills-dilaton theory. We present a mass formula for these stationary non-Abelian black holes, which also holds for Abelian black holes. The presence of the dilaton field allows for rotating black holes, which possess non-trivial electric and magnetic gauge fields, but don't carry a non-Abelian charge.

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Introduction Black holes in Einstein-Maxwell (EM) theory are uniquely characterized by their global charges: their mass M , their angular momentum J , their electric charge Q , and their magnetic charge P [1, 2]. For EM black holes remarkable relations between their horizon properties and their global charges hold [2], such as the Smarr formula [3],

$$M = 2TS + 2\Omega J + \Psi_{\text{el}}Q + \Psi_{\text{mag}}P, \quad (1)$$

where T represents the temperature of the black holes and S their entropy, Ω denotes their horizon angular velocity, and Ψ_{el} and Ψ_{mag} represent their horizon electrostatic and magnetic potential, respectively.

When non-Abelian fields are coupled to gravity, black hole solutions are no longer uniquely characterized by these global charges [4, 5]. Thus the EM “no-hair” theorem does not readily generalize to theories with non-Abelian gauge fields coupled to gravity, and neither does the mass formula, Eq. (1) [6, 7].

$SU(2)$ Einstein-Yang-Mills (EYM) theory, for instance, possesses sequences of static spherically and axially symmetric hairy black hole solutions, which carry non-Abelian magnetic fields but no non-Abelian charge [4, 8]; it further possesses sequences of rotating EYM black holes, which carry non-Abelian electric and magnetic fields, but only a non-Abelian electric charge [9, 10].

In many unified theories, including Kaluza-Klein theory and string theory, a scalar dilaton field arises naturally. When a dilaton field is coupled to EM theory, this has profound consequences for the black hole solutions [11, 12, 13]. Although uncharged Einstein-Maxwell-dilaton (EMD) black holes simply correspond to the EM black holes, charged EMD black hole solutions possess qualitatively new features. Charged static EMD black hole solutions, for instance, exist for arbitrarily small horizon size [11], and the surface gravity of ‘extremal’ solutions depends in an essential way on the dilaton coupling constant γ . Extremal charged rotating EMD black holes, known exactly only for Kaluza-Klein (KK) theory with $\gamma = \sqrt{3}$ [12, 13], can possess non-zero angular

momentum, while their event horizon has zero angular velocity [13].

The known EMD black hole solutions are still uniquely characterized by their mass, their angular momentum, and their electric and magnetic charge; and the mass formula Eq. (1) holds for the KK black hole solutions as well [13].

Here we consider stationary black hole solutions of $SU(2)$ Einstein-Yang-Mills-dilaton (EYMD) theory, and present a mass formula for these black holes. After showing that EMD black holes satisfy the mass formula

$$M = 2TS + 2\Omega J + \frac{D}{\gamma} + 2\Psi_{\text{el}}Q, \quad (2)$$

where the dilaton charge D enters instead of the magnetic charge P , we argue that this mass formula holds for all non-perturbatively known black hole solutions of $SU(2)$ EYMD theory [8, 14, 17]. The mass formula Eq. (2) generalizes the mass formula obtained previously for static purely magnetic non-Abelian black holes [8], $M = 2TS + D/\gamma$.

We then note that, while similar in many respects to the rotating EYM black holes [10], the EYMD black holes possess new features. In particular, we show that beyond a certain dilaton coupling strength γ , the presence of the dilaton allows for a new type of black hole: a rotating black hole which carries both electric and magnetic non-Abelian gauge fields but no non-Abelian charge.

Abelian mass formula Let us first consider the mass formula for EMD black hole solutions of the action

$$S = \int \left(\frac{R}{16\pi} + L_M \right) \sqrt{-g} d^4x \quad (3)$$

with matter Lagrangian

$$4\pi L_M = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4}e^{2\gamma\phi}F_{\mu\nu}F^{\mu\nu}. \quad (4)$$

We start from the general expression for the mass of

black holes [15]

$$M = 2TS + 2\Omega J_H - \frac{1}{4\pi} \int_{\Sigma} R_0^0 \sqrt{-g} dx d\theta d\varphi, \quad (5)$$

and express R_0^0 with help of the Einstein equations and the dilaton equation of motion,

$$R_0^0 = -\frac{1}{\gamma} \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} \partial^{\mu} \phi) + 2e^{2\gamma\phi} F_{0\alpha} F^{0\alpha}. \quad (6)$$

Evaluating the integral involving the dilaton d'Alembertian we obtain the dilaton term, D/γ , in the mass formula [8].

We then replace the horizon angular momentum J_H by the global angular momentum J [15],

$$J = J_H + \frac{1}{4\pi} \int_{\Sigma} e^{2\gamma\phi} F_{\varphi\alpha} F^{0\alpha} \sqrt{-g} dx d\theta d\varphi, \quad (7)$$

and obtain

$$M - 2TS - 2\Omega J - \frac{D}{\gamma} = -\frac{1}{2\pi} \int_{\Sigma} e^{2\gamma\phi} (F_{0\alpha} + \Omega F_{\varphi\alpha}) F^{0\alpha} \sqrt{-g} dx d\theta d\varphi. \quad (8)$$

To evaluate the remaining integral, we make the replacements $F_{\alpha 0} = \partial_{\alpha} A_0$ and $F_{\alpha\varphi} = \partial_{\alpha} A_{\varphi}$, and employ the gauge field equations of motion. The mass formula Eq. (2) then holds, provided

$$2\Psi_{\text{el}} Q = \frac{1}{2\pi} \int_{\Sigma} \partial_{\alpha} [(A_0 + \Omega A_{\varphi}) e^{2\gamma\phi} F^{0\alpha} \sqrt{-g}] dx d\theta d\varphi. \quad (9)$$

To show Eq. (9) we choose a gauge, where the gauge potential vanishes at infinity, and note that the electrostatic potential Ψ_{el}

$$\Psi_{\text{el}} = \chi^{\mu} A_{\mu} = A_0 + \Omega A_{\varphi}, \quad (10)$$

defined with Killing vector $\chi = \xi + \Omega\eta$ ($\xi = \partial_t$, $\eta = \partial_{\varphi}$) is constant at the horizon. Thus we obtain

$$2\Psi_{\text{el}} Q = -\Psi_{\text{el}} \frac{1}{2\pi} \int_H e^{2\gamma\phi} (*F_{\theta\varphi}) d\theta d\varphi, \quad (11)$$

which holds because the conserved charge \tilde{Q} [16],

$$\tilde{Q} = -\frac{1}{4\pi} \int_H e^{2\gamma\phi} (*F_{\theta\varphi}) d\theta d\varphi, \quad (12)$$

does not depend on the choice of 2-sphere, i.e., $\tilde{Q}(x_H) = \tilde{Q}(\infty) = \tilde{Q}$, and $\tilde{Q} = Q$ for $\phi(\infty) = 0$. When the Smarr formula holds [13], Eq. (2) implies $D/\gamma = \Psi_{\text{mag}} P - \Psi_{\text{el}} Q$.

Non-Abelian black holes We now turn to the non-Abelian black holes of the SU(2) EYMD action with matter Lagrangian

$$4\pi L_M = -\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} e^{2\gamma\phi} \text{Tr}(F_{\mu\nu} F^{\mu\nu}), \quad (13)$$

field strength tensor $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} + i[A_{\mu}, A_{\nu}]$ and gauge field $A_{\mu} = 1/2\tau^a A_{\mu}^a$.

For the metric we choose the stationary axially symmetric Lewis-Papapetrou metric in isotropic coordinates,

$$ds^2 = -f dt^2 + \frac{m}{f} (dx^2 + x^2 d\theta^2) + \frac{l}{f} x^2 \sin^2 \theta \left(d\varphi - \frac{\omega}{x} dt \right)^2. \quad (14)$$

For the gauge field we employ the ansatz [10],

$$A_{\mu} dx^{\mu} = \Psi dt + A_{\varphi} \left(d\varphi - \frac{\omega}{x} dt \right) + \left(\frac{H_1}{x} dx + (1 - H_2) d\theta \right) \frac{\tau_{\varphi}}{2}, \quad (15)$$

$$A_{\varphi} = -\sin \theta \left[H_3 \frac{\tau_x}{2} + (1 - H_4) \frac{\tau_{\theta}}{2} \right], \quad \Psi = B_1 \frac{\tau_x}{2} + B_2 \frac{\tau_{\theta}}{2}, \quad (16)$$

where the symbols τ_x , τ_{θ} and τ_{φ} denote the dot products of the Cartesian vector of Pauli matrices with the spherical spatial unit vectors. With respect to the residual gauge degree of freedom [8] we choose the gauge condition $x \partial_x H_1 - \partial_{\theta} H_2 = 0$ [10]. All functions depend only on x and θ . The above ansatz satisfies the Ricci circularity and Frobenius conditions [15].

The event horizon of stationary black holes resides at a surface of constant radial coordinate $x = x_H$, and is characterized by the condition $f(x_H) = 0$. At the horizon we impose the boundary conditions [10, 17] $f = m = l = 0$, $\omega = \omega_H = \Omega x_H$, $\partial_x \phi = 0$, $H_1 = 0$, $\partial_x H_2 = \partial_x H_3 = \partial_x H_4 = 0$, $B_1 - \Omega \cos \theta = 0$, $B_2 + \Omega \sin \theta = 0$, where Ω is the horizon angular velocity.

The boundary conditions at infinity, $f = m = l = 1$, $\omega = 0$, $\phi = 0$, $H_1 = H_3 = 0$, $H_2 = H_4 = \pm 1$, $B_1 = B_2 = 0$, ensure, that black holes are asymptotically flat and magnetically neutral. Axial symmetry and regularity impose the boundary conditions on the symmetry axis ($\theta = 0$), $\partial_{\theta} f = \partial_{\theta} l = \partial_{\theta} m = \partial_{\theta} \omega = 0$, $\partial_{\theta} \phi = 0$, $H_1 = H_3 = B_2 = 0$, $\partial_{\theta} H_2 = \partial_{\theta} H_4 = \partial_{\theta} B_1 = 0$, and agree with the boundary conditions on the $\theta = \pi/2$ -axis, except for $B_1 = 0$, $\partial_{\theta} B_2 = 0$.

To show the mass formula Eq. (2) for the non-Abelian black holes, we need to consider the asymptotic expansion for the metric, the gauge field and the dilaton functions [10, 17]. The mass M , the angular momentum $J = aM$, the non-Abelian electric charge Q (where $|Q|$ is the gauge invariant non-Abelian electric charge of Ref. [7]), and the dilaton charge D are obtained from the asymptotic expansion via

$$f \rightarrow 1 - \frac{2M}{x}, \quad \omega \rightarrow \frac{2J}{x^2}, \quad (17)$$

$$(\cos \theta B_1 + \sin \theta B_2) \rightarrow \frac{Q}{x}, \quad \phi \rightarrow -\frac{D}{x}. \quad (18)$$

We further need the horizon area A and the horizon temperature $T = \kappa_{\text{sg}}/2\pi$ with surface gravity κ_{sg} [15],

$$\kappa_{\text{sg}}^2 = -1/4 (\nabla_{\mu} \chi_{\nu}) (\nabla^{\mu} \chi^{\nu}). \quad (19)$$

To derive the mass formula for the non-Abelian black holes we first follow the arguments employed in the Abelian case. With the replacements $F_{\alpha 0} = D_\alpha A_0$ and $F_{\alpha\varphi} = D_\alpha(A_\varphi - u)$ [18], (where $u = \tau_z/2$, and $D_\alpha = \partial_\alpha + i[A_\alpha, \cdot]$), the mass formula holds, provided (see Eq. (9)),

$$2\Psi_{\text{el}}Q = \frac{1}{4\pi} \int_{\Sigma} \text{Tr} [D_\alpha (A_0 + \Omega(A_\varphi - u)) e^{2\gamma\phi} F^{0\alpha} \sqrt{-g}] dx d\theta d\varphi, \quad (20)$$

Since the trace of a commutator vanishes, we replace the gauge covariant derivative by the partial derivative, and again make use of the fact that the electrostatic potential is constant at the horizon, $\chi^\mu A_\mu = \Psi_H = \Omega u$, thus $\Psi_{\text{el}} = \Omega$ [17]. Hence the integral vanishes at the horizon, and we are left with the integral at infinity. Evaluating this integral with help of the asymptotic expansion [17] then yields the desired result. Thus the mass formula Eq. (2) holds for the non-Abelian black holes as well [17].

Numerical results We solve the set of eleven coupled non-linear elliptic partial differential equations numerically, subject to the above boundary conditions, employing compactified dimensionless coordinates, $\bar{x} = 1 - (x_H/x)$ [10].

Starting with a static spherically symmetric SU(2) EYMD black hole with horizon radius x_H , corresponding to $\omega_H = 0$, we choose a small but finite value for ω_H . The resulting rotating black hole then has non-trivial functions ω , H_1 , H_3 , B_1 , B_2 . By varying ω_H , while keeping the horizon parameter x_H and the dilaton coupling constant γ fixed, we obtain a set of rotating hairy black holes, analogous in many respects to the EYM black holes [10].

In Fig. 1 we display the specific angular momentum $a = J/M$, the non-Abelian electric charge Q , and the relative dilaton charge D/γ for black holes with horizon parameter $x_H = 0.1$ and dilaton coupling constant $\gamma = 1$ as functions of the mass M . As expected [10], these global charges are close to the corresponding global charges of the embedded Abelian solutions [19] with $Q = 0$ and $P = 1$.

In Fig. 2 we exhibit the global charges as functions of the dilaton coupling constant γ . With increasing γ the mass M and the relative dilaton charge D/γ decrease monotonically, the specific angular momentum a and the non-Abelian electric charge Q pass extrema [17]. Interestingly, the non-Abelian electric charge Q can change sign in the presence of the dilaton field.

Thus we observe the surprising feature that the non-Abelian charge Q of rotating EYMD black holes can vanish. Cuts through the parameter space of solutions with vanishing Q are exhibited in Fig. 3. Solutions with $Q = 0$ exist only above $\gamma_{\text{min}} \approx 1.15$. These $Q = 0$ EYMD black holes represent the first black hole solutions, which carry non-trivial non-Abelian electric and magnetic fields and no non-Abelian charge [20]. As a consequence, these special solutions do not exhibit the generic asymptotic non-integer power fall-off of the stationary non-Abelian gauge field solutions [10, 17].

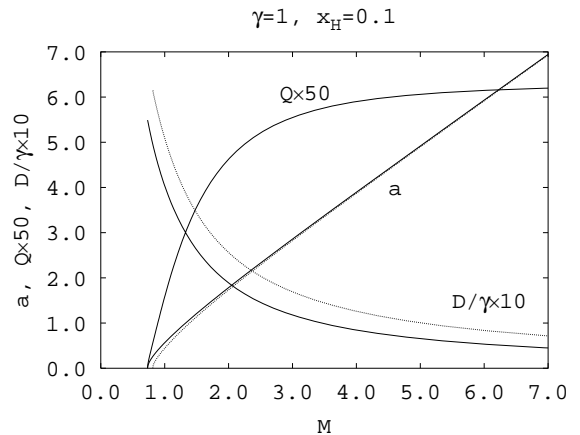


FIG. 1: The specific angular momentum a , the non-Abelian electric charge Q , and the relative dilaton charge D/γ are shown for the EYMD black hole solutions ($x_H = 0.1$, $\gamma = 1$) as functions of the mass M . Also shown are the corresponding properties of the embedded Abelian solutions with $Q = 0$ and $P = 1$ (dotted).

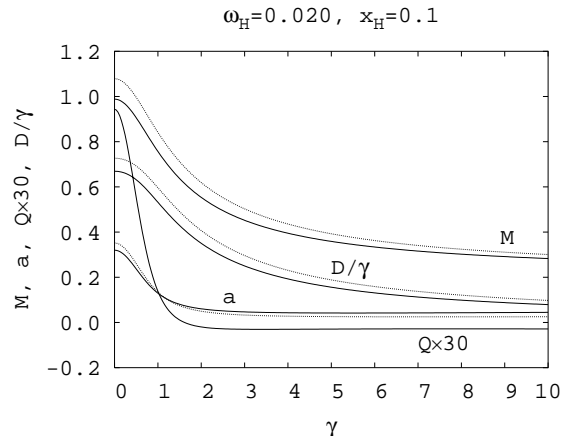


FIG. 2: The global charges (see Fig. 1) are shown as functions of the dilaton coupling constant γ ($x_H = 0.1$, $\omega_H = 0.02$).

Let us now turn to the horizon properties of the EYMD black holes. In Fig. 4 we show the area parameter $x_\Delta = \sqrt{A/4\pi}$, the temperature T , the deformation of the horizon (as quantified by the ratio of equatorial and polar circumferences) L_e/L_p , and the Gaussian curvature at the poles K . As for EM black holes [3], the Gaussian curvature of the horizon can become negative and the topology of the horizon is that of a 2-sphere.

The numerically constructed stationary axially symmetric EYMD black holes satisfy the mass formula, Eq. (2), with an accuracy of 10^{-3} . So do the numerically constructed EMD black holes. EYM black holes are included in the limit $\gamma \rightarrow 0$, since D/γ remains finite. Further details of the rotating non-Abelian black

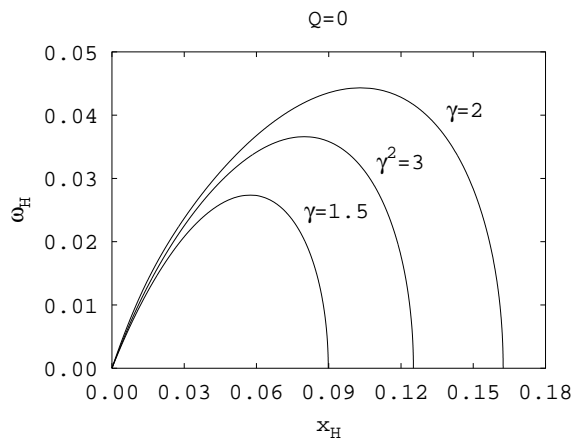


FIG. 3: Cuts through the parameter space of $Q = 0$ black hole solutions ($\gamma = 1.5, \sqrt{3}, 2$).

hole solutions will be given elsewhere [17].

Outlook The mass formula holds for the non-perturbatively known SU(2) EYMD black hole solutions. However, there may be further black hole solutions in SU(2) EYMD theory, with different boundary conditions and symmetries. For such black holes, the mass formula will have to be reconsidered.

Contrary to expectation, black holes in SU(2) EYMD

theory are not uniquely characterized by their mass M , their angular momentum J , their non-Abelian electric charge Q , and their dilaton charge D [17]. Thus a new uniqueness conjecture for non-Abelian black holes will have to include an additional charge [21].

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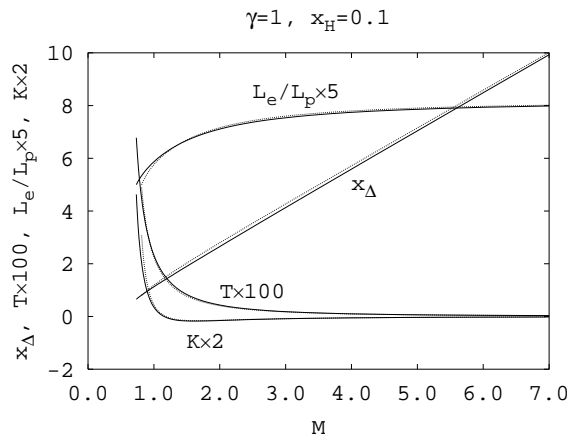


FIG. 4: Same as Fig. 1 for the area parameter x_Δ , the temperature T , the deformation of the horizon L_e/L_p , and the Gaussian curvature at the poles K .

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